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Admissible meromorphic solutions of algebraic differential equations[☆]

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ABSTRACT

In this paper, we extend Gackstatter–Laine's theorem to general algebraic differential equations, which is also the generalization of a theorem due to Wittich.

1. Introduction and main results

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$, the first and second main theorems etc., (see e.g. [1–4]). The notation $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of r of finite linear measure.

In general, a nonlinear algebraic differential equation is of the form:

$$P\left(z, f, f', \ldots, f^{(n)}\right) = 0, \tag{1}$$

where P is a polynomial in f and its derivatives with meromorphic coefficients. One can rewrite Eq. (1) in the form:

$$\sum_{\lambda \in I} \alpha_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n} = 0, \tag{2}$$

where I is a finite set of multi-indices $(\lambda_0, \lambda_1, \ldots, \lambda_n) = \lambda$ and $\alpha_\lambda(z)$ is a meromorphic function. We define a differential monomial in f as

$$M_\lambda[z,f] = \alpha_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n}.$$

The degree γ_{M_λ} and the weight Γ_{M_λ} of M_λ are defined by

$$\gamma_{M_\lambda} := \lambda_0 + \lambda_1 + \cdots + \lambda_n, \qquad \Gamma_{M_\lambda} := \lambda_0 + 2\lambda_1 + \cdots + (n+1)\lambda_n.$$

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Thus the left hand side of (2) can be expressed as a finite sum of differential monomials and which will be called a differential polynomial in f, i.e.,

$$P[z,f]=P\left(z,f,f',\ldots,f^{(n)}\right)=\sum_{\lambda\in I}M_\lambda[z,f].$$

The degree γ_P and the weight Γ_P of P are defined by

$$\gamma_P=\max_{\lambda\in I}\gamma_{M_\lambda},\qquad \Gamma_P=\max_{\lambda\in I}\Gamma_{M_\lambda}.$$

We say that the term $M_\lambda[z,f]$ is a dominant term of $P[z,f]$ if $\gamma_{M_\lambda}=\gamma_P$. Obviously, a differential polynomial may have more than one dominant term. A meromorphic solution f of Eq. (2) is called admissible, if $T(r,\alpha_\lambda)=S(r,f)$ holds for all coefficients $\alpha_\lambda(z)$, $\lambda\in I$.

In 1980, Gackstatter and Laine [5] considered the special algebraic differential equation of the following form,

$$(f')^n=Q(z,f),$$

where $Q(z,f)$ is a polynomial in f with meromorphic coefficients, and conjectured that it does not possess any admissible solution when $q:=\deg_f Q(z,f)\le n-1$.

This conjecture attracts many researchers' interest, see e.g. [6–11]. In 1990, He and Laine [6] gave a positive answer to the above conjecture. One year later, Ishizaki [7] proved the following more general result.

Theorem A. *The differential equation*

$$P(z,f')=Q(z,f),$$

where $P(z,f')$, resp. $Q(z,f)$, is a polynomial in f', resp. in f, with meromorphic coefficients such that $1\le q:=\deg_f Q(z,f)\le p-1:=\deg_{f'}P(z,f')-1$, admits no admissible solutions.

For a general algebraic differential equation, Wittich [12] gave a classic result as follows.

Theorem B. *If the algebraic differential equation*

$$P[z,f]=0,$$

where $P[z,f]$ is a differential polynomial in f with polynomial coefficients, has only one dominant term, then the equation has no transcendental entire solutions.

Now we give our main result, which is an extending result of Theorems A and B.

Theorem. *If the algebraic differential equation*

$$P[z,f]=0,\tag{3}$$

where $P[z,f]$ is a differential polynomial in f with meromorphic coefficients, has only one dominant term, then Eq. (3) has no admissible transcendental meromorphic solutions satisfying $N(r,f)=S(r,f)$.

The following two examples show that conditions $P[z,f]$ has only one dominant term and $N(r,f)=S(r,f)$ in theorem cannot be dropped.

Example 1. The following differential equation

$$(f')^2+\frac{2}{z}ff'+\left(1+\frac{1}{z^2}\right)f^2-\frac{1}{z^2}=0,\tag{4}$$

has an admissible transcendental meromorphic solution $f(z)=\frac{\cos z}{z}$ satisfying $N(r,f)=S(r,f)$. However, Eq. (4) has three dominant terms.

Example 2. The meromorphic function $f(z)=\tan(z^2)$ satisfies the following algebraic differential equation

$$(f')^2f''-4z(f')^3f-\frac{1}{z}(f')^3+f''=8z^2f^3+2f^2+8z^2f+2.\tag{5}$$

Eq. (5) has only one dominant term, but the counting function $N(r,f)=T(r,f)+S(r,f)$.

The following result is the extension of Theorem A.

Corollary. *The differential equation*

$$P(z,f^{(k)})=Q(z,f),\tag{6}$$

where $P(z,f^{(k)})$, resp. $Q(z,f)$, is a polynomial in $f^{(k)}$, resp. in f, with meromorphic coefficients such that $q:=\deg_f Q(z,f)\le p-1:=\deg_{f^{(k)}}P(z,f^{(k)})-1$ and $k\ge 1$ is a positive integer, has no admissible transcendental meromorphic solutions.

2. Some lemmas

The following lemmas will be needed in the proofs of our results.

Lemma 1 (*See [3]*). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f)=\frac{P(z,f)}{Q(z,f)}=\frac{\sum_{i=0}^{p}a_i(z)f^i}{\sum_{j=0}^{q}b_j(z)f^j},$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy

$$\begin{cases}T(r,a_i)=S(r,f), & i=0,1,\ldots,p,\\ T(r,b_j)=S(r,f), & j=0,1,\ldots,q,\end{cases}$$

we have

$$T(r,R(z,f))=\max\{p,q\}\cdot T(r,f)+S(r,f).$$

Lemma 2 (*See [7]*). *Let $f(z)$ be admissible relative to the coefficients of $P(z,f)=a_0(z)+a_1(z)f+\cdots+a_n(z)f^n$. Then*

$$N(r,P(z,f))=nN(r,f)+S(r,f).$$

Lemma 3 (*See [4]*). *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. Then*

$$T(r,f)<O\{T(2r,f')+\log r\}$$

as $r\to\infty$.

Lemma 4. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. Then*

$$T(r,f)<O\left\{T\left(r,f^{(k)}\right)+\log\frac{r}{2}\right\}\le O(T(r,f))+S(r,f) \tag{7}$$

holds for all $k\ge 1$ as $r\to\infty$.

Proof. Since $T(r,f)$ is a convex function of $\log r$, we have the following inequality

$$\frac{T(r,f)-T(r_0,f)}{\log r-\log r_0}\le\frac{T\left(\frac{r}{2},f\right)-T(r_0,f)}{\log\frac{r}{2}-\log r_0}$$

for all $r\ge r_0$. That is

$$T(r,f)-T(r_0,f)\le\frac{\log r-\log r_0}{\log r-\log r_0-\log 2}T\left(\frac{r}{2},f\right)-\frac{\log r-\log r_0}{\log r-\log r_0-\log 2}T(r_0,f)$$

holds for all $r\ge r_0$. Thus we obtain that, for any $\varepsilon>0$,

$$T(r,f)\le(1+\varepsilon)T\left(\frac{r}{2},f\right)+O(1) \tag{8}$$

as $r\to\infty$.

By Lemma 3, we have

$$T\left(\frac{r}{2},f\right)<O\left\{T(r,f')+\log\frac{r}{2}\right\} \tag{9}$$

as $r\to\infty$.

Therefore, the inequality

$$T(r,f)<O\left\{T(r,f')+\log\frac{r}{2}\right\} \tag{10}$$

holds as $r\to\infty$ by (8) and (9).

According to (10), we can easily get that

$$T(r,f)<O\left\{T\left(r,f^{(k)}\right)+\log\frac{r}{2}\right\} \tag{11}$$

holds for all $k\ge 1$ as $r\to\infty$.

On the other hand,

$$T\left(r,f^{(k)}\right) = m\left(r,f^{(k)}\right)+N\left(r,f^{(k)}\right)$$
$$\le m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+(k+1)N(r,f)$$
$$\le (k+1)T(r,f)+S(r,f). \tag{12}$$

Combining (11) and (12), we obtain the inequality (7). □

Lemma 5 (See [3]). *Let $f(z)$ be a transcendental meromorphic solution of*

$$f^nP[z,f]=Q[z,f],$$

where $P[z,f]$ and $Q[z,f]$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda|\lambda\in I\}$, such that $m(r,a_\lambda)=S(r,f)$ for all $\lambda\in I$. If the total degree of $Q[z,f]$ as a polynomial in f and its derivatives is $\le n$, then

$$m(r,P[z,f])=S(r,f).$$

3. Proofs of theorem and corollary

Proof of Theorem. We rewrite Eq. (3) in the following form:

$$P(z,f,f'\dots,f^{(n)}) = P(z,f,0,\dots,0)+\sum_{\lambda\in J}\alpha_\lambda(z)f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}$$
$$= -Q(z,f)+\Omega[z,f]=0,$$

where $\Omega[z,f]$, resp. $Q(z,f)$, is a differential polynomial, resp. a polynomial in f, with meromorphic coefficients, $J\subseteq I$ is a finite index set and $\lambda=(\lambda_0,\lambda_1,\lambda_2,\dots,\lambda_n)\ne(\lambda_0,0,0,\dots,0)$ if $\lambda\in J$. Obviously, $\Omega(z,f,0,\dots,0)=\Omega(z,0,\dots,0)\equiv 0$.

Thus, Eq. (3) can be changed into the following form:

$$\Omega[z,f]=Q(z,f), \tag{13}$$

where

$$\Omega[z,f]=\sum_{\lambda\in J}\alpha_\lambda(z)f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n},\quad \gamma_\Omega\ge 1,$$
$$Q(z,f)=\beta_q(z)f^q+\beta_{q-1}(z)f^{q-1}+\cdots+\beta_0(z),\quad q\ne\gamma_\Omega,$$

$\beta_q(z)\not\equiv 0$, and $T(r,\alpha_\lambda)=S(r,f)$, $T(r,\beta_j)=S(r,f)$ for $\lambda\in J$, $j=0,\dots,q$.

Next, we discuss four cases separately.

Case 1: $q\ge\gamma_\Omega+1$. We rewrite Eq. (13) in the following form:

$$\Omega_1[z,f]:=\Omega[z,f]-(\beta_{q-1}(z)f^{q-1}+\cdots+\beta_0(z))=\beta_q(z)f^q,$$

where $\Omega_1[z,f]$ is a differential polynomial in f with meromorphic coefficients and the total degree of $\Omega_1[z,f]$ is $\le q-1$. Thus by Lemma 5, we obtain $m(r,f)=S(r,f)$. Therefore, $T(r,f)=m(r,f)+N(r,f)=S(r,f)$, this is a contradiction. So we may assume that $q\le\gamma_\Omega-1$ in the next three cases.

Case 2: $\gamma_\Omega\ge 3$ and $q\ge 1$.

For any integer $k\ge 1$, since $N(r,f)=S(r,f)$, we have

$$N(r,f^{(k)})\le (k+1)N(r,f)=S(r,f). \tag{14}$$

Next, we have that

$$\begin{cases} m\left(r,\dfrac{1}{f-\tau}\right)=S(r,f)\\ Q(z,\tau)\not\equiv 0\end{cases} \tag{15}$$

hold simultaneously outside of a finite exceptional set of τ values. In fact, substituting $f=\tau+\frac{1}{u}$ into (13), and noting that $\Omega[z,f]$ has only one dominant term, $\Omega(z,0,\dots,0)=\Omega(z,f,0,\dots,0)\equiv 0$ and $q<\gamma_\Omega$, we have

$$(\beta_q\tau^q+\cdots+\beta_1\tau+\beta_0)u^{\Gamma_\Omega}=S_{\Gamma_\Omega-1}(u), \tag{16}$$

where $S_{\Gamma_\Omega - 1}(u)$ is a differential polynomial in u of total degree at most $\Gamma_\Omega - 1$. We can choose $\tau \in \mathbb{C}$ such that $\beta_q \tau^q + \cdots + \beta_1 \tau + \beta_0 \not\equiv 0$, in fact, there are at most q complex numbers τ such that $\beta_q \tau^q + \cdots + \beta_1 \tau + \beta_0 \equiv 0$. Thus, u is an admissible transcendental meromorphic solution of (16). By Lemma 5, we have $m(r, u) = S(r, u)$. Therefore, (15) follows.

Take now distinct complex constants $\tau_1, \ldots, \tau_{\gamma_\Omega}$ to satisfy (15). The constants $\tau_1, \ldots, \tau_{\gamma_\Omega}$ will be specified later on. It is immediate to deduce, for each $\gamma_{M_\lambda} \leq \gamma_\Omega (\lambda \in J)$, that

$$m\left(r, \frac{M_\lambda[z,f]}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right) = m\left(r, \frac{\alpha_\lambda(z) f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right) = S(r,f). \tag{17}$$

In fact, we may write

$$\frac{\alpha_\lambda(z) f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}$$
$$= \alpha_\lambda(z)\frac{f}{f-\tau_1}\cdots\frac{f}{f-\tau_{\lambda_0}}\frac{f'}{f-\tau_{\lambda_0+1}}\cdots\frac{f'}{f-\tau_{\lambda_0+\lambda_1}}\cdots\frac{f^{(n)}}{f-\tau_{\gamma_{M_\lambda}}}\cdots\frac{1}{f-\tau_{\gamma_{M_\lambda}+1}}\cdots\frac{1}{f-\tau_{\gamma_\Omega}}.$$

Since

$$m\left(r, \frac{f}{f-\tau_\mu}\right) = m\left(r, 1+\frac{\tau_\mu}{f-\tau_\mu}\right) \leq m\left(r, \frac{1}{f-\tau_\mu}\right) + O(1) = S(r,f), \tag{18}$$

for $\mu = 1, \ldots, \lambda_0$. Therefore, (17) follows from (15), (18) and the lemma of logarithmic derivative.

Now we define

$$F(z,\tau_j) := \frac{\Omega[z,f] - Q(z,\tau_j)}{f-\tau_j}, \quad j = 1, \ldots, \gamma_\Omega. \tag{19}$$

Obviously, $\Omega[z,f] - Q(z,\tau_j) = Q(z,f) - Q(z,\tau_j)$ is divisible by $f - \tau_j$, therefore (19) and Lemma 2 implies

$$N\left(r, F(z,\tau_j)\right) = S(r,f), \quad j = 1, \ldots, \gamma_\Omega. \tag{20}$$

We define

$$h(z) := \sum_{j=1}^{\gamma_\Omega} A_j F(z,\tau_j) = \Omega[z,f]\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j} - \sum_{j=1}^{\gamma_\Omega}\frac{A_j Q(z,\tau_j)}{f-\tau_j}, \tag{21}$$

for some complex constants $A_1, \ldots, A_{\gamma_\Omega}$, which will be specified immediately. It follows from (20) that $N(r,h) = S(r,f)$. By (15), we obtain

$$m\left(r, \sum_{j=1}^{\gamma_\Omega}\frac{A_j Q(z,\tau_j)}{f-\tau_j}\right) = S(r,f). \tag{22}$$

Now we choose complex constants $A_1, \ldots, A_{\gamma_\Omega}$ to satisfy

$$\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j} = \frac{A}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}, \tag{23}$$

where $A = \sum_{j=1}^{\gamma_\Omega}\left(A_j(-1)^{\gamma_\Omega - 1}\prod_{i=1,i\neq j}^{\gamma_\Omega}\tau_i\right) \neq 0$ is a non-zero constant. In fact, this choice is regarded as a nontrivial solution of the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{\gamma_\Omega}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(\gamma_\Omega-2)} & \sigma_2^{(\gamma_\Omega-2)} & \cdots & \sigma_{\gamma_\Omega}^{(\gamma_\Omega-2)} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{\gamma_\Omega}\end{bmatrix} = 0, \tag{24}$$

where σ_n^m is a fundamental symmetric expression of $\tau_j (1 \leq j \leq \gamma_\Omega, j \neq n)$ of degree m.

It follows from (23) and (17) that

$$m\left(r, \Omega[z,f]\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j}\right) = m\left(r, \frac{A\Omega[z,f]}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right) = S(r,f). \tag{25}$$

Combining (22) and (25), we get that $m(r,h) = S(r,f)$ and so $T(r,h) = S(r,f)$.

Assuming now that h doesn't vanish identically, we obtain by (21) and (23) that

$$A\Omega[z,f] = h(z)\prod_{j=1}^{\gamma_\Omega}(f-\tau_j) + \sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right). \tag{26}$$

Namely,

$$AQ(z,f) = h(z)\prod_{j=1}^{\gamma_\Omega}(f-\tau_j) + \sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right). \tag{27}$$

From (27) and Lemma 1, we conclude that $qT(r,f) = \gamma_\Omega T(r,f) + S(r,f)$. Thus $\gamma_\Omega T(r,f) = qT(r,f) + S(r,f) \leq (\gamma_\Omega - 1)T(r,f) + S(r,f)$. Therefore we get the contradiction $T(r,f) = S(r,f)$.

Next we may assume that h vanishes identically. Using (21) and (23) again, we have

$$A\Omega[z,f] = \sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right).$$

From (13), we obtain the identity

$$\frac{1}{A}\sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right) = \sum_{j=0}^{q}\beta_j(z)f^j. \tag{28}$$

If $\sum_{j=1}^{\gamma_\Omega} A_jQ(z,\tau_j) \not\equiv 0$, then $q = \gamma_\Omega - 1$, otherwise we have $T(r,f) = S(r,f)$. Therefore, by comparing the coefficients of $f^{\gamma_\Omega - 1}$ on both sides of (28), we obtain the following equation

$$t_1(z)A_1 + t_2(z)A_2 + \cdots + t_{\gamma_\Omega}(z)A_{\gamma_\Omega} = 0,$$

where $t_j(z) = Q(z,\tau_j) - (-1)^{\gamma_\Omega - 1}\beta_{\gamma_\Omega - 1}(z)\prod_{i=1,i\neq j}^{\gamma_\Omega}\tau_i$. Since $\gamma_\Omega \geq 3$ and $\beta_q(z) \not\equiv 0$, we may choose $\tau_j (j = 1,\ldots,\gamma_\Omega)$ such that the determinant of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{\gamma_\Omega}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(\gamma_\Omega - 2)} & \sigma_2^{(\gamma_\Omega - 2)} & \cdots & \sigma_{\gamma_\Omega}^{(\gamma_\Omega - 2)} \\ t_1(z) & t_2(z) & \cdots & t_{\gamma_\Omega}(z) \end{bmatrix}$$

is not equal to zero identically, thus we obtain $A_1 = \cdots = A_{\gamma_\Omega} = 0$, which contradicts our assumption.

If $\sum_{j=1}^{\gamma_\Omega} A_jQ(z,\tau_j) \equiv 0$, but $\sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)(-1)\sum_{i=1,i\neq j}^{\gamma_\Omega}\tau_i\right) \not\equiv 0$, then $q = \gamma_\Omega - 2$, otherwise we have $T(r,f) = S(r,f)$. By comparing the coefficients of $f^{\gamma_\Omega - 2}$ on both sides of (28), we obtain the following equation

$$s_1(z)A_1 + s_2(z)A_2 + \cdots + s_{\gamma_\Omega}(z)A_{\gamma_\Omega} = 0,$$

where $s_j(z) = Q(z,\tau_j)(-1)\sum_{i=1,i\neq j}^{\gamma_\Omega}\tau_i - (-1)^{\gamma_\Omega - 1}\beta_{\gamma_\Omega - 2}(z)\prod_{i=1,i\neq j}^{\gamma_\Omega}\tau_i$. Since $\gamma_\Omega \geq 3$ and $\beta_q(z) \not\equiv 0$, we may choose $\tau_j (j = 1,\ldots,\gamma_\Omega)$ such that the determinant of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{\gamma_\Omega}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(\gamma_\Omega - 2)} & \sigma_2^{(\gamma_\Omega - 2)} & \cdots & \sigma_{\gamma_\Omega}^{(\gamma_\Omega - 2)} \\ s_1(z) & s_2(z) & \cdots & s_{\gamma_\Omega}(z) \end{bmatrix}$$

is not equal to zero identically, thus we have $A_1 = \cdots = A_{\gamma_\Omega} = 0$, which is a contradiction.

By the similar discussion, we can obtain contradictions for $q = \gamma_\Omega - 3, \dots, 1$. Thus, we complete the proof of Case 2.

Case 3: $\gamma_\Omega = 2$ and $q = 1$.

In this case, the Eq. (13) becomes one of the following two forms:

$$ff^{(k)} + D_1(f) = \beta_1(z)f + \beta_0(z), \tag{29}$$

or

$$f^{(m)}f^{(n)} + D_2(f) = \beta_1^*(z)f + \beta_0^*(z), \tag{30}$$

where $k, m, n \geq 1$ are integers, $\beta_1(z), \beta_1^*(z) \not\equiv 0$, $D_1(f)$ and $D_2(f)$ are finite sums in the derivatives of f with meromorphic coefficients and $\gamma_{D_1}, \gamma_{D_2} \leq 1$.

Squaring both sides of Eqs. (29) and (30), we have

$$f^2(f^{(k)})^2 + D_1^*(f) = \beta_1^2 f^2 + 2\beta_1\beta_0 f + \beta_0^2, \tag{31}$$

or

$$(f^{(m)})^2(f^{(n)})^2 + D_2^*(f) = \beta_1^{*2} f^2 + 2\beta_1^*\beta_0^* f + \beta_0^{*2}, \tag{32}$$

where $D_1^*(f), D_2^*(f)$ are polynomial in f and its derivatives with meromorphic coefficients, and $\gamma_{D_1^*}, \gamma_{D_2^*} \leq 3$. Thus case 3 changes into case 2 and satisfies the assumption in the theorem (noting that we have not assumed $P[z, f]$ is irreducible in the proof of case 2). According to case 2, Eqs. (31) and (32) have no admissible transcendental meromorphic solutions. Therefore, Eqs. (29) and (30) also have no admissible transcendental meromorphic solutions.

Case 4: $q = 0$. In this case, Eq. (13) reduces into the following form:

$$\Omega[z, f] = \beta_0(z). \tag{33}$$

Since $q \leq \gamma_\Omega - 1$, we have $\gamma_\Omega \geq 1$. If $\gamma_\Omega = 1$, then Eq. (13) must be the following form:

$$\alpha(z)f^{(k)} = \beta_0(z), \tag{34}$$

where $\alpha(z) \not\equiv 0$ is a meromorphic function satisfying $T(r, \alpha(z)) = S(r, f)$ and $k \geq 1$ is an integer. Thus, we have $T(r, f^{(k)}) = S(r, f)$. Hence it follows from Lemma 4 that $T(r, f) = S(r, f)$, this is a contradiction. Therefore, $\gamma_\Omega \geq 2$.

Subcase 1: $\beta_0(z) \not\equiv 0$. Multiplying f on both sides of Eq. (33), then we have the following new equation:

$$\Delta[z, f] := \sum_{\lambda \in J} \alpha_\lambda(z) f^{\lambda_0+1}(f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n} = \beta_0(z)f. \tag{35}$$

Since $\gamma_\Delta \geq 3$, this is case 2. Thus (35) has no admissible transcendental meromorphic solutions with $N(r, f) = S(r, f)$. Therefore, Eq. (33) also has no admissible transcendental meromorphic solutions with $N(r, f) = S(r, f)$.

Subcase 2: $\beta_0(z) \equiv 0$. We may make a transformation $f = g + \gamma(z)$, where $\gamma(z)$ is a meromorphic function satisfying $T(r, \gamma(z)) = S(r, f)$. Substituting it into (33), we obtain the following equation:

$$N[z, g] := \sum_{\mu \in K} a_\mu(z) g^{\mu_0}(g')^{\mu_1} \cdots (g^{(n)})^{\mu_n} = b(z), \tag{36}$$

where $N[z, g]$ is a differential polynomial in g with meromorphic coefficients $a_\mu(z)$ satisfying $T(r, a_\mu(z)) = S(r, g)$, $N(z, 0, \dots, 0) \equiv 0$, K is a finite set of multi-indices $(\mu_0, \mu_1, \dots, \mu_n) = \mu$, $b(z) = -(\gamma(z))^{\lambda_0}(\gamma'(z))^{\lambda_1} \cdots (\gamma^{(n)}(z))^{\lambda_n}$ is a meromorphic function. We can choose $\gamma(z)$ such that $b(z) \not\equiv 0$. Obviously, $T(r, b(z)) = S(r, f) = S(r, g)$, $\gamma_N = \gamma_\Omega \geq 2$ and (36) has only one dominant term. Thus, this case changes into one of the cases of case 2, case 3 and subcase 1. Thus (36) has no admissible transcendental meromorphic solutions with $N(r, g) = S(r, g)$. Hence we conclude that (33) has no admissible transcendental meromorphic solutions with $N(r, f) = S(r, f)$. The proof of the theorem is completed. □

Proof of Corollary. By the theorem, we only need to prove that $N(r, f) = S(r, f)$. By Lemma 2 and the fact that $q \leq p - 1$, we immediately obtain that

$$\begin{aligned}
(p-1)N(r, f) + S(r, f) &\geq qN(r, f) + S(r, f) \\
&= N(r, Q(z, f)) \\
&= N(r, P(z, f^{(k)})) + S(r, f) \\
&= pN(r, f^{(k)}) + S(r, f) \\
&\geq pN(r, f) + S(r, f).
\end{aligned}$$

This is a contradiction unless $N(r, f) = S(r, f)$. The proof is completed. □

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