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ON MEROMORPHIC SOLUTIONS OF CERTAIN TYPE OF NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We consider meromorphic solutions of non-linear differential equation of the form

$$f^n + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},$$

where $Q_d(z, f)$ is a differential polynomial in f of degree $d \leq n - 2$ with rational functions as its coefficients, p_1, p_2 are rational functions and α_1, α_2 are polynomials. More precisely and mainly we have shown the conditions concerning α_1'/α_2' that will ensure the existence and forms of the possible meromorphic solutions of the above equation. These results have extended and improved some known results obtained most recently.

1. Introduction and main results

In studying differential equations in the complex plane **C**, it's always an interesting and quite difficult problem to prove the existence or uniqueness of the entire or meromorphic solution of a given differential equation, particularly for a non-linear ones. Since 1970's , Nevanlinna's value distribution theory (particularly Clunie type of lemmas relating equations involving differential polynomials) have been used or utilized by the second author of the paper and his co-workers (see, e.g., [9, 11, 12, 13]) to tackle the non-linear differential equations of the form

$$f^n + P_d(z, f) = h,$$

where $P_d(z, f)$ denotes a polynomial in f and its derivatives with a total degree $d \leq n - 1$, with small functions of f as the coefficients, and h is a given entire or meromorphic function. Moreover, $P_d(z, f)$ is called an algebraic differential polynomial in f, if all its coefficients are polynomials in z. We assume that the reader is familiar with the standard notations in the Nevanlinna theory (see[2, 4]) and its associated standard notations, such as

$$T(r, f),\ m(r, f),\ N(r, f),\ \overline{N}(r, f), \cdots .$$

We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o\{T(r, f)\}, \quad \text{as } r \to \infty,$$

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possibly outside of a set E with finite linear measure, not necessarily the same at each occurrence.

Recently, it is shown in [13] that the equation $4f^3(z) + 3f''(z) = -\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$, $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. More recently, the following two results have been obtained:

Theorem A. [9] *Let $n \geq 4$ be an integer and $P_d(z, f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leq n-3$ with small functions of f as the coefficients. If $p_1(z), p_2(z)$ are two nonzero polynomials and α_1, α_2 are two nonzero constants such that $\frac{\alpha_1}{\alpha_2}$ is not rational, then the equation*

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}$$

does not have any transcendental entire solution.

Theorem B. [8] *Let $n \geq 2$ be an integer, $P_d(z, f)$ be an algebraic differential polynomial in $f(z)$ of degree $d \leq n-2$ with small functions of f as the coefficients, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is a transcendental meromorphic solution of the following equation*

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1}$$

and satisfying $N(r, f) = S(r, f)$, then one of the following holds:

(1) $f(z) = c_0 + c_1 e^{\frac{\alpha_1 z}{n}}$;
(2) $f(z) = c_0 + c_2 e^{\frac{\alpha_2 z}{n}}$;
(3) $f(z) = c_1 e^{\frac{\alpha_1 z}{n}} + c_2 e^{\frac{\alpha_2 z}{n}}$, *and* $\alpha_1 + \alpha_2 = 0$,

where c_0 is a small function of $f(z)$ and c_1, c_2 are constants satisfying $c_1^n = p_1$, $c_2^n = p_2$.

Now we shall extend the above results by considering that h is a meromorphic function of finite (integer) order and improve the results of Theorems A and B, as well as that of [5, 6] and [15].

Theorem 1. *Let $n \geq 3$ and $Q_d(z, f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that p_1, p_2 are rational functions and α_1, α_2 are polynomials. If $d \leq n-2$, the following differential equation*

$$f^n + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{2}$$

admits a meromorphic function f with finitely many poles. Then $\frac{\alpha_1'}{\alpha_2'}$ is a rational number. Furthermore, only one of the following four cases holds:

(1) $f(z) = q(z)e^{P(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = 1$, *where $q(z)$ is a rational function and $P(z)$ is a polynomial with $nP'(z) = \alpha_1' = \alpha_2'$;*
(2) $f(z) = q(z)e^{P(z)}$ *and either* $\frac{\alpha_1'}{\alpha_2'} = \frac{k}{n}$ *or* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$, *where $q(z)$ is a rational function, k is an integer with $1 \leq k \leq d$ and $P(z)$ is a polynomial with $nP'(z) = \alpha_1'$ or $nP'(z) = \alpha_2'$;*
(3) *f satisfies the first order linear differential equation $f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) f + \psi$ and $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$ or f satisfies the first order linear differential equation $f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1'\right) f + \psi$ and $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1}$, where ψ is a rational function;*

(4) $f(z) = \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z)e^{-\beta_1(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = -1$, *where* γ_1, γ_2 *are rational functions and* $\beta_1(z)$ *is a polynomial with* $n\beta_1' = \alpha_1'$ *or* $n\beta_1' = \alpha_2'$.

Remark. The four cases in the theorem exist. For instance, $f = e^z + z + 1$ solves the following non-linear differential equation $f^3 - 2(z+1)^2 f'' - (z+1)^2 f = e^{3z} + 3(z+1)e^{2z}$. This example shows the case (3) in the theorem certainly exists.

Corollary 1. *Let* $n \geq 3$ *and* $Q_d(z, f)$ *be a differential polynomial in* f *of degree* d *with rational functions as its coefficients. Suppose that* p_1, p_2 *are rational functions and* α_1, α_2 *are constants. If* $d \leq n-2$, *the following differential equation*

$$f^n + Q_d(z, f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}, \tag{3}$$

admits a meromorphic function f *with finitely many poles. Then* $\frac{\alpha_1}{\alpha_2}$ *is a rational number. Furthermore, only one of the following four cases holds:*

(1) $\frac{\alpha_1}{\alpha_2} = 1$ *and* $f(z) = q(z)e^{\frac{\alpha_1 z}{n}}$, *where* $q(z)^n = p_1(z) + p_2(z)$ *is a rational function;*

(2) $\frac{\alpha_1}{\alpha_2} = \frac{n}{k}$ *for some* $1 \leq k \leq d$ *and* $f(z) = q(z)e^{\frac{\alpha_1 z}{n}}$, *where* $q(z)^n = p_1(z)$ *or* $\frac{\alpha_1}{\alpha_2} = \frac{k}{n}$ *for some* $1 \leq k \leq d$ *and* $f(z) = q(z)e^{\frac{\alpha_2 z}{n}}$, *where* $q(z)^n = p_2(z)$;

(3) $\frac{\alpha_1}{\alpha_2} = \frac{n-1}{n}$ *and* f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2\right) f + \psi$ *or* $\dfrac{\alpha_1}{\alpha_2} = \frac{n}{n-1}$ *and* f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1\right) f + \psi$, *where* ψ *is a rational function;*

(4) $\alpha_1 + \alpha_2 = 0$ *and* $f(z) = q_1(z)e^{\frac{\alpha_1 z}{n}} + q_2(z)e^{-\frac{\alpha_1 z}{n}}$, *where* $q_1(z)^n = p_1(z)$ *and* $q_2(z)^n = p_2(z)$.

Theorem 2. *Let* $n \geq 3$ *and* $Q_d(z, f)$ *be a differential polynomial in* f *of degree* d *with rational functions as its coefficients. Suppose that* R, p_1, p_2 *are rational functions and* α_1, α_2 *are polynomials. If* $d \leq n-2$ *and the following differential equation*

$$f^n + R(z)f^{n-1} + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{4}$$

admits a meromorphic function f *with finitely many poles. Then* $\frac{\alpha_1'}{\alpha_2'}$ *is a rational number. Furthermore, only one of the following four cases holds:*

(1) $f(z) = -\frac{R(z)}{n} + q(z)e^{P(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = 1$, *where* $q(z)$ *is a rational function with and* $P(z)$ *is a polynomial with* $nP'(z) = \alpha_1' = \alpha_2'$;

(2) $f(z) = -\frac{R(z)}{n} + q(z)e^{P(z)}$ *and either* $\frac{\alpha_1'}{\alpha_2'} = \frac{k}{n}$ *or* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$, *where* $q(z)$ *is a rational function,* k *is an integer with* $1 \leq k \leq d$ *and* $P(z)$ *is a polynomial with* $nP'(z) = \alpha_1'$ *or* $nP'(z) = \alpha_2'$;

(3) f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) f + \psi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$ *or* f *satisfies the first order linear differential equation* $f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1'\right) f + \psi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1}$, *where* ψ *is a rational function;*

(4) $f(z) = -\frac{R(z)}{n} + \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z)e^{-\beta_1(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = -1$, *where* γ_1, γ_2 *are rational functions and* $\beta_1(z)$ *is a polynomial with* $n\beta_1' = \alpha_1'$ *or* $n\beta_1' = \alpha_2'$.

Corollary 2. *Let* $n \geq 3$ *and* $Q_d(z, f)$ *be a differential polynomial in* f *of degree* d *with rational functions as its coefficients. Suppose that* R, p_1, p_2 *are rational functions* *and α_1, α_2 are constants. If $d \le n-2$, the following differential equation*

$$f^n + R(z)f^{n-1} + Q_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}, \tag{5}$$

admits a meromorphic function f with finitely many poles. Then $\frac{\alpha_1}{\alpha_2}$ is a rational number. Furthermore, only one of the following four cases holds:

(1) *$\frac{\alpha_1}{\alpha_2} = 1$ and $f(z) = \frac{R(z)}{n} + q(z)e^{\frac{\alpha_1 z}{n}}$, where $q(z)^n = p_1(z) + p_2(z)$ is a rational function;*

(2) *$\frac{\alpha_1}{\alpha_2} = \frac{n}{k}$ for some $1 \le k \le d$ and $f(z) = \frac{R(z)}{n} + q(z)e^{\frac{\alpha_1 z}{n}}$, where $q(z)^n = p_1(z)$ or $\frac{\alpha_1}{\alpha_2} = \frac{k}{n}$ for some $1 \le k \le d$ and $f(z) = \frac{R(z)}{n} + q(z)e^{\frac{\alpha_2 z}{n}}$, where $q(z)^n = p_2(z)$;*

(3) *$\frac{\alpha_1}{\alpha_2} = \frac{n-1}{n}$ and f satisfies the first order linear differential equation $f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2\right) f + \psi$ or $\frac{\alpha_1}{\alpha_2} = \frac{n}{n-1}$ and f satisfies the first order linear differential equation $f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1\right) f + \psi$, where ψ is a rational function;*

(4) *$\alpha_1 + \alpha_2 = 0$ and $f(z) = -\frac{R(z)}{n} + q_1(z)e^{\frac{\alpha_1 z}{n}} + q_2(z)e^{-\frac{\alpha_1 z}{n}}$, where $q_1(z)^n = p_1(z)$ and $q_2(z)^n = p_2(z)$.*

Let $R(z) = \frac{P(z)}{Q(z)} \not\equiv 0$ be a rational function, where $P(z), Q(z)$ are co-prime polynomials. We define the degree of R at ∞ $\deg_\infty R = \deg P - \deg Q$. If $R(z) \equiv 0$, we define $\deg_\infty R = -\infty$. Thus, if $R(z)$ is a non-zero polynomial, then $\deg_\infty R = \deg R$. It is easy to check that $\deg_\infty \frac{R'}{R} = -1$ if $R(z)$ is a non-constant rational function. Hence, $\lim\limits_{z\to\infty} \frac{R'(z)}{R(z)} = 0$ if $R(z)$ is a nonzero rational function. If R_1, R_2 are two nonzero rational functions, then $\deg_\infty \frac{R_1}{R_2} = \deg_\infty R_1 - \deg_\infty R_2$.

2. Lemmas

Lemma 1. [4, p. 51] *Let f be a transcendental entire function, and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta} \tag{6}$$

holds. Then there exists a set F in $\mathbf{R}^+$ of finite logarithmic measure, i.e., $\int_F 1/t\,dt < +\infty$ such that

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m (1+o(1))f(z) \tag{7}$$

holds whenever m is a fixed nonnegative integer and $r \notin F$.

Lemma 2. [14] *Let $f(z)$ be a nonconstant meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r), \quad r \to \infty,$$

if f is of finite order, and

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r,f))), \quad r \to \infty,$$

possibly outside a set E of r with finite linear measure if $f(z)$ is of infinite order.

The following can be easily derived from the proof of the Clunie lemma, see e.g. [1, 4].

Lemma 3. *Let $f(z)$ be meromorphic and transcendental function in the plane and satisfy*

$$f^n(z)P(f) = Q(f),$$

where $P(f), Q(f)$ are differential polynomials in $f(z)$ with rational functions as the coefficients and the degree of $Q(f)$ is at most n, then

$$m(r, P(f)) = O(\log r), \quad r \to \infty,$$

if f is of finite order, and

$$m(r, P(f)) = O(\log(rT(r, f))), \quad r \to \infty,$$

possibly outside a set E of r with finite linear measure if $f(z)$ is of infinite order.

Lemma 4. [14], [3, Lemma 5.1] *Let $a_j(z)$ be entire funciton of finite order $\leq \rho$. Let $g_j(z)$ be entire and $g_k(z) - g_j(z)$, $j \neq k$, be a transcendental entire function or polynomial of degree greater than ρ. Then*

$$\sum_{j=1}^{n} a_j(z) e^{g_j(z)} = a_0(z)$$

holds only when

$$a_0(z) = a_1(z) = \cdots = a_n(z) \equiv 0.$$

The following lemma is crucial to the proofs of our results.

Lemma 5. *Let q_1, q_2, q_3, a be rational functions and $q_3 a \not\equiv 0$. If the differential equation*

$$q_1(z) f^2 + q_2(z) f f' + q_3(z) f'^2 = a(z), \tag{8}$$

admits a transcendental meromorphic solution, then

(i) *any meromorphic solution of (8) must be of finite order, and*

(ii) *the following identity holds:*

$$q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q_3' \equiv 0,$$

and any transcendental meromorphic solution f of the equation (8) satisfies the following linear second order differential equation

$$f'' = \left(\frac{a'}{2a} - \frac{q_3'}{2q_3} - \frac{q_2}{2q_3}\right) f' - \frac{1}{q_2}\left(q_1' - q_1\frac{a'}{a}\right) f.$$

Furthermore, if $q_2^2 - 4q_1q_3 \not\equiv 0$ and $\deg_\infty q_2 \geq \deg_\infty q_3$, then the differential equation (8) has no transcendental meromorphic solution.

Proof. Let f be a transcendental meromorphic solution of the equation (8). If z_0 is a pole of f, which is not a zero and pole of q_1, q_2 and q_3, then z_0 is a pole of a. Therefore, f has only finitely many poles. Thus there is a polynomial $P(z)$ such that $f(z)P(z) = g(z)$ is a transcendental entire function. Let $|g(z_0)| = M(r, g)$, $|z_0| = r$. Then, by Lemma 1, we have

$$\frac{f'(z_0)}{f(z_0)} = \frac{g'(z_0)}{g(z_0)} - \frac{P'(z_0)}{P(z_0)} = \frac{\nu(r, g)}{z_0}(1 + o(1)), \quad r \notin F,$$

where F is a set of a finite logarithmic measure. Then, from the equation (8), we have

$$q_3(z_0)\left(\frac{\nu(r,g)}{z_0}(1+o(1))\right)^2+q_2(z_0)\frac{\nu(r,g)}{z_0}(1+o(1))+q_1(z_0)=\frac{a(z_0)P(z_0)^2}{g(z_0)^2}.$$

It follows for sufficiently large r that

$$\nu(r,g)\le A\left(\left|\frac{q_2(z_0)z_0}{q_3(z_0)}\right|+\left|\frac{q_1(z_0)z_0^2}{q_3(z_0)\nu(r,g)}\right|\right)\le A\left(\left|\frac{q_2(z_0)z_0}{q_3(z_0)}\right|+\left|\frac{q_1(z_0)z_0^2}{q_3(z_0)}\right|\right).$$

Hence, g has finite order, so does f. We rewrite the equation (8) as

$$\frac{1}{f^2}=\frac{q_1}{a}+\frac{q_2}{a}\frac{f'}{f}+\frac{q_3}{a}\left(\frac{f'}{f}\right)^2. \tag{9}$$

According to Lemma 2 and the above equation , it follows that $m\big(r,\frac{1}{f}\big)=O(\log r)$ and $T(r,f)=N\big(r,\frac{1}{f}\big)+O(\log r)$. Hence, f has infinitely many zeros. Further, a zero of f is simple if it is not a zero of $a(z)$ and a pole of q_1,q_2,q_3. Differentiating (8) yields

$$q_1'f^2+(2q_1+q_2')ff'+q_2ff''+(q_2+q_3')(f')^2+2q_3f'f''=a'. \tag{10}$$

Assume z_0 is a zero of f which is not the pole of q_1,q_2,q_3 and a, also is not the zero of a. Then from (8) and (10), we have $q_3(z_0)f'(z_0)^2=a(z_0)$ and $(q_2(z_0)+q_3'(z_0))f'(z_0)^2+2q_3(z_0)f'(z_0)f''(z_0)=a'(z_0)$, which implies that z_0 is a zero of $(a'q_3-aq_2-aq_3')f'-2aq_3f''$. Let

$$R(z)=\frac{(a'q_3-aq_2-aq_3')f'-2aq_3f''}{f}.$$

Then $R(z)$ has only finitely many poles and it follows from Lemma 2 that $m(r,R)=O(\log r)$. Hence $R(z)$ is a rational function. It follows that

$$f''=\frac{a'q_3-aq_2-aq_3'}{2aq_3}f'-\frac{R}{2aq_3}f. \tag{11}$$

By substituting the above equation into (10), we obtain

$$\left(q_1'-\frac{Rq_2}{2aq_3}\right)f^2+\left(2q_1+q_2'+\frac{q_2(a'q_3-aq_2-aq_3')}{2aq_3}-\frac{R}{a}\right)ff'+q_3\frac{a'}{a}(f')^2=a'. \tag{12}$$

It follows from (8) and (12) that

$$A(z)f+B(z)f'=0, \tag{13}$$

where

$$A(z)=q_1'-\frac{Rq_2}{2aq_3}-q_1\frac{a'}{a}\quad\text{and}\quad B(z)=2q_1+q_2'-\frac{q_2a'}{2a}-\frac{q_2^2+q_2q_3'}{2q_3}-\frac{R}{a}.$$

Noting $A(z),B(z)$ are rational functions and f has infinitely many simple zeros, we have $B(z)\equiv0$, and hence $A(z)\equiv0$. By eliminating R from the above two equations, we can get, as asserted

$$q_3(q_2^2-4q_1q_3)\frac{a'}{a}+q_2(q_2^2-4q_1q_3)-q_3(q_2^2-4q_1q_3)'+(q_2^2-4q_1q_3)q_3'\equiv0,$$

and

$$f''=\left(\frac{a'}{2a}-\frac{q_3'}{2q_3}-\frac{q_2}{2q_3}\right)f'-\frac{1}{q_2}\left(q_1'-q_1\frac{a'}{a}\right)f.$$

Finally, if $q_2^2 - 4q_1q_3 \not\equiv 0$, then the above equation can be written as

$$\frac{q_2}{q_3} = \frac{(q_2^2 - 4q_1q_3)'}{q_2^2 - 4q_1q_3} - \frac{a'}{a} - \frac{q_3'}{q_3}. \tag{14}$$

If $\deg_\infty \frac{q_2}{q_3} \geq 0$, then the left side of the equation (14) goes to infinity or a non-zero number as $z \to \infty$. However, the right side of the equation (14) goes to zero as $z \to \infty$. This contradiction yields the conclusion that the equation (8) has no transcendental meromorphic solution. This completes the proof of the lemma. $\square$

Lemma 6. *Let $n \geq 2$ be an integer and $P_d(z, f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leq n-1$ with small functions of f as the coefficients. If $p_1(z), p_2(z)$ are small functions of f and α_1, α_2 are two nonconstant polynomials. If f is a meromorphic solution of the equation*

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$$

and $N(r, f) = S(r, f)$, then f is of finite order.

Proof. Clearly, any meromorphic function satisfying the equation in the lemma must be transcendental. Denote $k_1 = \deg \alpha_1, k_2 = \deg \alpha_2$ and $k = \max\{k_1, k_2\}$. By Clunie Lemma and $N(r, f) = S(r, f)$, we have

$$\begin{aligned} nT(r, f) &= m(r, f^n) + S(r, f) \\ &\leq T(r, p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) + m(r, P_d(z, f)) + S(r, f) \\ &\leq Ar^k + dT(r, f) + S(r, f). \end{aligned}$$

Thus $(n-d)T(r, f) \leq Ar^k + S(r, f)$ and f is of finite order. $\square$

3. Proofs of the theorems

3.1. Proof of Theorem 1. Let f be a meromorphic solution with finitely many poles of the equation (2). It follows from Lemma 6 that the order of f is finite. Denote $g(z) = Q_d(z, f)$. Then

$$nf^{n-1}f' + g' = (p_1' + \alpha_1'p_1)e^{\alpha_1(z)} + (p_2' + \alpha_2'p_2)e^{\alpha_2(z)}. \tag{15}$$

By eliminating e^{α_2} from the equation (3) and (15), we have

$$(p_2' + \alpha_2'p_2)f^n - np_2f^{n-1}f' + (p_2' + \alpha_2'p_2)g - p_2g' = A_1(z)e^{\alpha_1(z)}, \tag{16}$$

where $A_1(z) = p_1(p_2' + \alpha_2'p_2) - p_2(p_1' + \alpha_1'p_1)$. If $A_1(z) \equiv 0$, then $\alpha_2' - \alpha_1' = \frac{p_1'}{p_1} - \frac{p_2'}{p_2}$. Thus $\alpha_2' - \alpha_1' \equiv 0$ and the equation (16) becomes

$$(p_2' + \alpha_2'p_2)f^n - np_2f^{n-1}f' = -(p_2' + \alpha_2'p_2)g + p_2g'.$$

It follows from Lemma 3 that

$$(p_2' + \alpha_2'p_2)f^2 - np_2ff' = \psi_1(z)$$

and

$$(p_2' + \alpha_2'p_2)f - np_2f' = \psi_2(z),$$

where $\psi_1(z), \psi_2(z)$ are rational functions. If $\psi_2(z) \not\equiv 0$, then $f(z) = \frac{\psi_1(z)}{\psi_2(z)}$ is a rational function, which is a contradiction. Hence,

$$(p_2' + \alpha_2'p_2)f - np_2f' = 0.$$

By solving the above equation, we obtain $f(z)^n = Cp_2e^{\alpha_2(z)}$. This is the case (1). Now, we assume $A_1(z) \not\equiv 0$. Denote

$$F(z) = \frac{1}{A_1(z)}\left((p_2' + \alpha_2'p_2)f^n - np_2f^{n-1}f' + (p_2' + \alpha_2'p_2)g - p_2g'\right),$$

then we have

$$Br^{k_1} = T(r, e^{\alpha_1}) + o(1) = T(r, F) + o(1) \leq nT(r, f) + S(r, f), \tag{17}$$

where $k_1 = \deg \alpha_1$ and B is a positive constant. By differentiating the equation (16), we have

$$\begin{aligned} &(p_2' + \alpha_2'p_2)'f^n + n\alpha_2'p_2f^{n-1}f' - n(n-1)p_2f^{n-2}f'^2 - np_2f^{n-1}f'' \\ &+ (p_2' + \alpha_2'p_2)'g + \alpha_2'p_2g' - p_2g'' = (A_1' + \alpha_1'A_1)e^{\alpha_1(z)}. \end{aligned} \tag{18}$$

By eliminating $e^{\alpha_1(z)}$ from the equation (16) and (18), we have

$$(h_1(z)f^2 + h_2(z)ff' + h_3(z)f'^2 + h_4(z)ff'')f^{n-2} = Q_d^*(z, f),$$

where

$$\begin{aligned} Q_d^*(z, f) = &(p_2' + \alpha_2'p_2)'A_1g + \alpha_2'p_2A_1g' - p_2A_1g'' \\ &- (p_2' + \alpha_2'p_2)(A_1' + \alpha_1'A_1)g + p_2(A_1' + \alpha_1'A_1)g' \end{aligned}$$

is a differential polynomial of f with degree $d \leq n-2$ and rational functions as coefficients and

$$\begin{aligned} h_1 &= (p_2' + \alpha_2'p_2)(A_1' + \alpha_1'A_1) - (p_2' + \alpha_2'p_2)'A_1, \\ h_2 &= -n(\alpha_1' + \alpha_2')p_2A_1 - np_2A_1', \\ h_3 &= n(n-1)p_2A_1, \\ h_4 &= np_2A_1, \end{aligned}$$

are rational functions. It follows from Lemma 3 that

$$h_1(z)f^2 + h_2(z)ff' + h_3(z)f'^2 + h_4(z)ff'' = a(z), \tag{19}$$

where $a(z)$ is a rational function. Next, we discuss two cases.

Case 1. $a(z) \equiv 0$. Then the equation (19) can be rewritten as

$$h_1(z)f^2 = -(h_2(z)ff' + h_3(z)f'^2 + h_4(z)ff'').$$

Let z_0 be a zero of f with multiplicity k, but no zero and pole of h_1, h_2, h_3, h_4. Then z_0 is a zero with multiplicity $2k$ of left side of the above equation and a zero with at most multiplicity $2k-1$ of right side of the above equation. This contradiction lead to that f has at most finitely many zeros. Thus, $f(z) = q(z)e^{P(z)}$, where $q(z)$ is a rational function and $P(z)$ is a polynomial. Substituting $f(z) = q(z)e^{P(z)}$ into the equation (2) yields

$$q(z)^ne^{nP(z)} + \sum_{k=0}^{d} a_k(z)e^{kP(z)} = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},$$

where $a_k(z)(k = 0, 1 \cdots d)$ are rational functions. If $\alpha_1'(z) \equiv \alpha_2'(z)$, then $\alpha_2(z) = \alpha_1(z) + C$ and it follows from Lemma 4 that $a_k(z) \equiv 0$ for all $k(1 \leq k \leq d)$ and $nP'(z) = \alpha_1'(z)$. If $\alpha_1'(z) \not\equiv \alpha_2'(z)$, it follows from Lemma 4 that $a_k(z) \not\equiv 0$ for some $k(1 \leq k \leq d)$ and $a_j(z) \equiv 0$ when $j \neq k(0 \leq j \leq d)$. Furthermore either $q(z)^n =$

$B_1p_1(z), nP(z) = \alpha_1(z)+C_1, kP(z) = \alpha_2(z)+C_2$ or $q(z)^n = B_2p_2(z), nP(z) = \alpha_2(z)+C_2, kP(z) = \alpha_1(z) + C_1$, where B_1, B_2, C_1, C_2 are constants and $B_1e^{C_1} = B_2e^{C_2} = 1$. Hence, $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$ or $\frac{k}{n}$.

Case 2. $a(z) \not\equiv 0$. If f has only finitely many zeros, then by the similar argument in Case 1, we have $f(z) = q(z)e^{P(z)}$, where $q(z)$ is a rational function and $P(z)$ is a polynomial, and one of the following two subcases holds: (i) $\alpha_1' \equiv \alpha_2'$; (ii) either $q(z)^n = B_1p_1(z), \frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$ or $q(z)^n = B_2p_2(z), \frac{\alpha_1'}{\alpha_2'} = \frac{k}{n}$.

Now we assume that f has infinitely many zeros. By differentiating (19), we get

$$h_1'f^2+(2h_1+h_2')ff'+(h_2+h_3')f'^2+(h_2+h_4')ff''+(2h_3+h_4)f'f''+h_4ff''' = a'(z) \tag{20}$$

Suppose z_0 is a zero of f that is not the zero and pole of h_1, h_2, h_3, h_4 and $a(z)$. Then from (19) and (20), we have

$$h_3(z_0)f'(z_0)^2 = a(z_0),$$

and

$$(h_2(z_0) + h_3'(z_0))\, f'(z_0)^2 + (2h_3(z_0) + h_4(z_0))f'(z_0)f''(z_0) = a'(z_0),$$

which implies that $f'(z_0) \neq 0$ and z_0 is a simple zero of f, and further z_0 is a zero of $(a'h_3 - ah_2 - ah_3')f' - (2ah_3 + ah_4)f''$. Let

$$\beta = \frac{(a'h_3 - ah_2 - ah_3')f' - (2ah_3 + ah_4)f''}{f}.$$

Then we have $T(r, \beta) = O(\log r)$, thus β is a rational function. It follows that

$$f'' = \frac{a'h_3 - ah_2 - ah_3'}{2ah_3 + ah_4}f' - \frac{\beta}{2ah_3 + ah_4}f. \tag{21}$$

By substituting the above equation into (19), we have

$$q_1(z)f^2 + q_2(z)ff' + q_3(z)(f')^2 = a(z), \tag{22}$$

where

$$q_1(z) = h_1 - \frac{\beta}{2ah_3 + ah_4}h_4, \quad q_2(z) = h_2 + \frac{a'h_3 - ah_2 - ah_3'}{2ah_3 + ah_4}h_4 \quad \text{and} \quad q_3(z) = h_3$$

are rational function. Furthermore,

$$\frac{q_2(z)}{q_3(z)} = -\frac{2}{(2n-1)}(\alpha_1' + \alpha_2') - \frac{3}{2n-1}\frac{A_1'}{A_1} + \frac{1}{2n-1}\frac{a'}{a} - \frac{1}{2n-1}\frac{p_2'}{p_2}. \tag{23}$$

If $\alpha_1' + \alpha_2' \not\equiv 0$, then $\deg_\infty \frac{q_2(z)}{q_3(z)} \geq 0$. If $q_2^2 - 4q_1q_3 \not\equiv 0$, then by Lemma 5, at this case the equation (22) has no meromorphic solution. If $q_2^2 - 4q_1q_3 \equiv 0$, then by Lemma 5, f satisfies the following differential equation

$$f'' = \left(\frac{a'}{2a} - \frac{q_3'}{2q_3} - \frac{q_2}{2q_3}\right) f' - \frac{q_2}{4q_3}\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right) f. \tag{24}$$

It follows from the equations (21) and (24) that

$$\frac{\beta}{2ah_3 + ah_4} = \frac{q_2}{4q_3}\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right).$$

By $q_2^2 - 4q_1q_3 \equiv 0$, we also have

$$\frac{\beta}{2ah_3 + ah_4} = \frac{h_1}{h_4} - \frac{1}{4h_3h_4}\left(h_2 + \frac{a'h_3 - ah_2 - ah_3'}{2ah_3 + ah_4}h_4\right)^2.$$

The above two equations yield that

$$\left(-\frac{1}{2(2n-1)}(\alpha_1' + \alpha_2') - \frac{3}{4(2n-1)}\frac{A_1'}{A_1} + \frac{1}{4(2n-1)}\frac{a'}{a} - \frac{1}{4(2n-1)}\frac{(p_2)'}{p_2}\right)\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right)$$
$$= \frac{1}{n}\left(\alpha_1'\alpha_2' + \frac{p_2'}{p_2}\frac{A_1'}{A_1} + \alpha_2'\frac{A_1'}{A_1} + \alpha_1'\frac{p_2'}{p_2} - \frac{p_2''}{p_2} - \alpha_2'' - \alpha_2'\frac{p_2'}{p_2}\right)$$
$$- \frac{n-1}{4}\left(-\frac{2}{(2n-1)}(\alpha_1' + \alpha_2') - \frac{3}{2n-1}\frac{A_1'}{A_1} + \frac{1}{2n-1}\frac{a'}{a} - \frac{1}{2n-1}\frac{p_2'}{p_2}\right)^2.$$

This yields that $\deg_\infty(\alpha_1' + \alpha_2')^2 = \deg_\infty \alpha_1'\alpha_2' \geq 0$ and $\lim_{z\to\infty} \frac{\alpha_1'\alpha_2'}{(\alpha_1'+\alpha_2')^2} = \frac{n(n-1)}{(2n-1)^2}$. Hence, we have $\deg \alpha_1 = \deg \alpha_2$. Let $\alpha_1(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ and $\alpha_2(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0$, where $a_m b_m \neq 0, m \geq 1$. It follows from $\lim_{z\to\infty} \frac{\alpha_1'\alpha_2'}{(\alpha_1'+\alpha_2')^2} = \frac{n(n-1)}{(2n-1)^2}$ that $\lim_{z\to\infty} \frac{a_m b_m}{(a_m+b_m)^2} = \frac{n(n-1)}{(2n-1)^2}$. Thus $\frac{a_m}{b_m} = \frac{n-1}{n}$ or $\frac{n}{n-1}$.

First, we discuss the case $\frac{a_m}{b_m} = \frac{n-1}{n}$. The equation (16) can be written as

$$(25) \qquad (p_2' + \alpha_2' p_2)f^n - np_2 f^{n-1}f' + (p_2' + \alpha_2' p_2)g - p_2 g' = \phi_1(z)e^{a_m z^m},$$

where $\phi_1(z) = (p_1(p_2' + \alpha_2' p_2) - p_2(p_1' + \alpha_1' p_1))\, e^{a_{m-1}z^{m-1}+\cdots+a_0}$. It follows from (17) that $T(r, \phi_1) = Cr^{k-1} = S(r, f)$. Similarly, we have

$$(26) \qquad (p_1' + \alpha_1' p_1)f^n - np_1 f^{n-1}f' + (p_1' + \alpha_1' p_1)g - p_1 g' = \phi_2(z)e^{b_m z^m},$$

where $\phi_2(z) = -\left(p_1(p_2' + \alpha_2' p_2) - p_2(p_1' + \alpha_1' p_1)\right) e^{b_{m-1}z^{m-1}+\cdots+b_0}$ and $T(r, \phi_2) = S(r, f)$. It follows from (25) and (26) that

$$(p_2' + \alpha_2' p_2)f^n - np_2 f^{n-1}f' + (p_2' + \alpha_2' p_2)g - p_2 g'$$
$$= \phi_1(z)\left(\frac{(p_1' + \alpha_1' p_1)f^n - np_1 f^{n-1}f' + (p_1' + \alpha_1' p_1)g - p_1 g'}{\phi_2(z)}\right)^{\frac{n-1}{n}}.$$

Thus

$$(27) \qquad \begin{aligned} (p_2' + \alpha_2' p_2)\, f - np_2 f' &= -\frac{(p_2' + \alpha_2' p_2)g - p_2 g'}{f^{n-1}} \\ &\quad + \phi_1(z)\left(\frac{(p_1' + \alpha_1' p_1)f^n - np_1 f^{n-1}f' + (p_1' + \alpha_1' p_1)g - p_1 g'}{\phi_2(z) f^n}\right)^{\frac{n-1}{n}}. \end{aligned}$$

Since f has only finitely many poles, we have

$$T(r, (p_2' + \alpha_2' p_2)f - np_2 f') = m(r, (p_2' + \alpha_2' p_2)f - np_2 f') + S(r, f)$$
$$= \frac{1}{2\pi}\int_{E_1} \log^+ \left|(p_2'(re^{i\theta}) + \alpha_2'(re^{i\theta})p_2(re^{i\theta}))f(re^{i\theta}) - np_2(re^{i\theta})f'(re^{i\theta}))\right| \mathrm{d}\theta +$$
$$\frac{1}{2\pi}\int_{E_2} \log^+ \left|(p_2'(re^{i\theta}) + \alpha_2'(re^{i\theta})p_2(re^{i\theta}))f(re^{i\theta}) - np_2(re^{i\theta})f'(re^{i\theta}))\right| \mathrm{d}\theta + S(r, f),$$

where $E_1 = \{\theta\colon |f(re^{i\theta})| \le 1\}$, $E_2 = \{\theta\colon |f(re^{i\theta})| \ge 1\}$. Now

$$\begin{aligned}
&\frac{1}{2\pi}\int_{E_1} \log^+ \left|(p_2'(re^{i\theta}) + \alpha_2'(re^{i\theta})p_2(re^{i\theta}))f(re^{i\theta}) - np_2(re^{i\theta})f'(re^{i\theta}))\right| \mathrm{d}\theta \\
&\le \frac{1}{2\pi}\int_{E_1} \log^+ \left|f'(re^{i\theta})\right| \mathrm{d}\theta + O(\log r) \le \frac{1}{2\pi}\int_{E_1} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \mathrm{d}\theta + O(\log r) \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \mathrm{d}\theta + O(\log r) = S(r,f).
\end{aligned}$$

It follows from (27) that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{E_2} \log^+ \left|(p_2'(re^{i\theta}) + \alpha_2'(re^{i\theta})p_2(re^{i\theta}))f(re^{i\theta}) - np_2(re^{i\theta})f'(re^{i\theta}))\right| \mathrm{d}\theta \\
&\le \frac{1}{2\pi}\int_{E_2} \log^+ \left|\frac{(p_2' + \alpha_2'p_2)g - p_2g'}{f^{n-1}}(re^{i\theta})\right| \mathrm{d}\theta + \frac{1}{2\pi}\int_{E_2} \log^+ \left|\phi_1(re^{i\theta})\right| \mathrm{d}\theta \\
&\quad + \frac{n-1}{n}\frac{1}{2\pi}\int_{E_2} \log^+ \left|\frac{(p_1' + \alpha_1'p_1)f^n - np_1f^{n-1}f' + (p_1' + \alpha_1'p_1)g - p_1g'}{\phi_2 f^n}(re^{i\theta})\right| \mathrm{d}\theta \\
&= S(r,f).
\end{aligned}$$

Hence

$$T(r, (p_2' + \alpha_2'p_2)f - np_2f') = S(r,f).$$

Thus

$$(p_2' + \alpha_2'p_2)f - np_2f' = \phi_3(z), \tag{28}$$

where $T(r,\phi_3) = S(r,f)$. It follows from (28) that

$$f' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) f - \psi_3, \tag{29}$$

where $\psi_3 = \dfrac{\phi_3}{np_2}$ is a small meromorphic function of f. Differentiating (29), we obtain

$$f'' = \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) f' + \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right)' f - \psi_3'. \tag{30}$$

It follows from (24) and (30) that

$$\left(\frac{a'}{2a} - \frac{q_3'}{2q_3} - \frac{q_2}{2q_3} - \frac{1}{n}\frac{p_2'}{p_2} - \frac{1}{n}\alpha_2'\right) f' = \left[\frac{q_2}{4q_3}\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right) + \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right)'\right] f - \psi_3'. \tag{31}$$

By (29) and (31), we have

$$\left(\frac{a'}{2a} - \frac{q_3'}{2q_3} - \frac{q_2}{2q_3} - \frac{1}{n}\frac{p_2'}{p_2} - \frac{1}{n}\alpha_2'\right)\left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) = \frac{q_2}{4q_3}\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right) + \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right)'. \tag{32}$$

It follows from (23) and (32) that

$$\begin{aligned}
&\left(\frac{1}{2n-1}\alpha_1' - \frac{n-1}{n(2n-1)}\alpha_2' + \frac{n-1}{2n-1}\frac{a'}{a} - \frac{n-2}{2n-1}\frac{A_1'}{A_1} - \frac{n^2+n-1}{n(2n-1)}\frac{p_2'}{p_2}\right)\left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right) \\
&= \frac{1}{4}\left(-\frac{2}{(2n-1)}(\alpha_1' + \alpha_2') - \frac{3}{2n-1}\frac{A_1'}{A_1} + \frac{1}{2n-1}\frac{a'}{a} - \frac{1}{2n-1}\frac{p_2'}{p_2}\right)\left(\frac{q_1'}{q_1} - \frac{a'}{a}\right) \\
&\quad + \left(\frac{1}{n}\frac{p_2'}{p_2} + \frac{1}{n}\alpha_2'\right)'.
\end{aligned} \tag{33}$$

If $\frac{1}{2n-1}\alpha_1' - \frac{n-1}{n(2n-1)}\alpha_2' \not\equiv 0$, then denote $\frac{1}{2n-1}\alpha_1' - \frac{n-1}{n(2n-1)}\alpha_2' = c_k z^k + \cdots + c_0$, $k \geq 0$, $c_k \neq 0$. Dividing the both sides of (33) by z^{m+k-1} and taking limits as $z \to \infty$, we obtain an impossible equation $\frac{m}{n} c_k b_m = 0$. This yields that $\frac{1}{2n-1}\alpha_1' - \frac{n-1}{n(2n-1)}\alpha_2' \equiv 0$, i.e. $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$. It follows from (29), (31) and $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$ that

$$\frac{\psi_3'}{\psi_3} = \frac{n-1}{2n-1}\frac{a'}{a} - \frac{n-2}{2n-1}\frac{A_1'}{A_1} - \frac{n^2+n-1}{n(2n-1)}\frac{p_2'}{p_2}.$$

This equation yields that ψ_3 is a rational function. If $\frac{a_m}{b_m} = \frac{n}{n-1}$, by similar arguments, we have $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1}$ and

$$f' = \left(\frac{1}{n}\frac{p_1'}{p_1} + \frac{1}{n}\alpha_1'\right) f + \psi,$$

where ψ is a rational function.

If $\alpha_1' + \alpha_2' \equiv 0$, then $\alpha_2 = -\alpha_1 + C$, where C is constant, and the equation (2) becomes

$$f^n + Q_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_3(z)e^{-\alpha_1(z)}, \tag{34}$$

where $p_3(z) = e^C p_2(z)$. We now denote $Q_d(z, f)$ by $g(z)$. By differentiating the equation (34), we get

$$nf^{n-1}f' + g' = (p_1' + p_1\alpha_1')e^{\alpha_1(z)} + (p_3' - p_3\alpha_1')e^{-\alpha_1(z)}. \tag{35}$$

Eliminating $e^{\alpha_1(z)}$ and $e^{-\alpha_1(z)}$ respectively from the equations (34) and (35) yields

$$\begin{aligned} &(p_3' - p_3\alpha_1')f^n - np_3f^{n-1}f' + (p_3' - p_3\alpha_1')g - p_3g' \\ &= [p_1(p_3' - p_3\alpha_1') - p_3(p_1' + p_1\alpha_1')]e^{\alpha_1(z)}, \end{aligned} \tag{36}$$

and

$$\begin{aligned} &(p_1' + p_1\alpha_1')f^n - np_1f^{n-1}f' + (p_1' + p_1\alpha_1')g - p_1g' \\ &= [p_3(p_1' + p_1\alpha_1') - p_1(p_3' - p_3\alpha_1')]e^{-\alpha_1(z)}. \end{aligned} \tag{37}$$

It follows from the equations (36) and (37) that

$$[(p_3' - p_3\alpha_1')f - np_3f'][(p_1' + p_1\alpha_1')f - np_1f']f^{2n-2} + Q_{2n-2}(z, f) = -A(z)^2, \tag{38}$$

where

$$\begin{aligned} Q_{2n-2}(z, f) = &[(p_3' - p_3\alpha_1')f^n - np_3f^{n-1}f'][(p_1' + p_1\alpha_1')g - p_1g'] \\ &+ [(p_1' + p_1\alpha_1')f^n - np_1f^{n-1}f'][+(p_3' - p_3\alpha_1')g - p_3g'] \end{aligned}$$

is a differential polynomial of f with degree $\leq 2n-2$, with rational functions as coefficients, and $A(z) = [p_1(p_3' - p_3\alpha_1') - p_3(p_1' + p_1\alpha_1')]$ is a rational function. It follows from Lemma 3 again that

$$[(p_3' - p_3\alpha_1')f - np_3f'][(p_1' + p_1\alpha_1')f - np_1f'] = b(z), \tag{39}$$

where $b(z)$ is a rational function. Hence,

$$(p_3' - p_3\alpha_1')f - np_3f' = b_1(z)e^{\beta_1(z)}, \tag{40}$$

and

$$(p_1' + p_1\alpha_1')f - np_1f' = b_2(z)e^{-\beta_1(z)}, \tag{41}$$

where $b_1(z), b_2(z)$ are rational functions such that $b_1(z)b_2(z) = b(z)$ and $\beta_1(z)$ is a polynomial. The above two equations yield immediately that

$$f = \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z)e^{-\beta_1(z)},$$

where

$$\gamma_1 = \frac{p_1 b_1(z)}{p_1 p_3' - p_1' p_3 - 2p_1 p_3 \alpha_1'}, \quad \gamma_2 = \frac{-p_3 b_2(z)}{p_1 p_3' - p_1' p_3 - 2p_1 p_3 \alpha_1'}.$$

This also completes the proof of the theorem.

3.2. Proof of Theorem 2. Assume that f is a meromorphic solution with only finitely many poles of the equation (4). Let $g(z) = f(z) + \frac{R(z)}{n}$. Then, g is a transcendental meromorphic function with only finitely many poles and satisfies the following differential equation

$$f^n + Q^*_{n-2}(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{42}$$

where $Q^*_{n-2}(z, f)$ is a differential equation with degree $\leq n - 2$. The conclusions of the theorem follows immediately from Theorem 1.

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